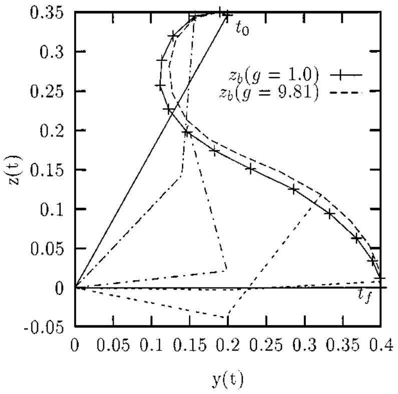

Fig. 2 Trajectories of the elbow's tip for g = 9.81 and 1.

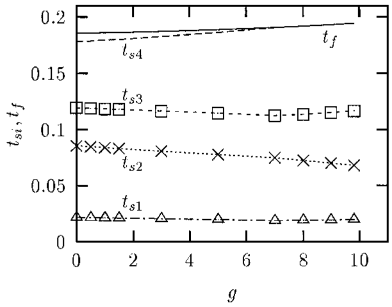

Fig. 3 Effects of g on final time t_f and switch times t_{si}.

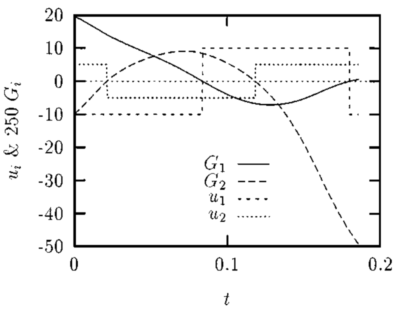

Fig. 4 Optimal control and switch functions for g = 1.

maneuver times. In the vertical plane, $t_f = 0.194386$ s and three switches are required to execute the maneuver, whereas in the almost horizontal plane, $t_f = 0.186025$ s (4.5% less) and four switches are required. Figure 3 indicates the effect of g on the final time t_f (solid line) and the switch times t_{si}. Note that the time-optimal maneuvers have four switches if $g \leq 7 \text{ m} \cdot \text{s}^{-2}$ and three switches if $g > 7 \text{ m} \cdot \text{s}^{-2}$. Also note that despite the downward motion of the links, the gravity increases the maneuver time. The switching patterns and the corresponding switch functions for $g = 1$ are shown in Fig. 4. For $g = 9.81$ the switching patterns are similar, with the exception that the last switch, t_{s4}, for u_1 disappears. For both cases (four switches and three switches) the torque at the elbow switches twice, whereas the torque at the shoulder switches once for the vertical motion and twice for $g = 1$.

IV. Conclusions

The results indicate that gravity always increases the duration of time-optimal rest-to-rest maneuvers, independently of whether the links movement is against or with the force of gravity. It seems to be due to the fact that during the maneuver the control torques switch between the upper bound and lower bound whereas the gravity force must remain the same. For the downward motion presented, even if gravity is beneficial in the accelerating phase, when starting from rest, it appears to be more detrimental in the decelerating phase that brings the manipulator back to rest. For an upward motion the effects are similar.

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Pinch Points of Debris from a Satellite Breakup

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Introduction

DURING the first few orbits after a satellite breakup, it is well known[1] that the behavior of the fragments is characterized by "pinch points," where the fragment dispersion vanishes in either one or two directions, leaving the fragments scattered (to first order) in a single plane or along a single line.

These pinch points are generally considered to occur after each half-orbit. The out-of-plane motion (at least for slow dispersion from a near-circular orbit) is a simple sinusoid, so that all fragments return to the original orbit plane after each half-revolution. In addition, the in-plane motion is such that all fragments pass through the breakup point after each full revolution. Hence, the half-orbit pinch point is a planar one and the full-orbit one is a linear one.

Our purpose here is to point out that there exists an additional pinch point during each orbit (except for the first one after breakup), whose existence does not seem to have been appreciated previously in the orbital-debris literature. The phenomenon is identical, however, to one previously shown to exist in the context of interplanetary guidance.

Analysis

For analysis purposes, only a circular parent orbit is considered. For small perturbations it has been shown[2–4] that the in-plane position vector $\boldsymbol{r}(t) = [x(t) \;\; y(t)]^T$ of a fragment (relative to the undisturbed orbit) at any time t after the breakup is given by

$$\boldsymbol{r}(t) = \Phi(t)\boldsymbol{v}_0 \tag{1}$$

where $\boldsymbol{v}_0$ is its breakup velocity vector $[\dot{x}_0 \;\; \dot{y}_0]^T$ and $\Phi(t)$ is a transition matrix given by

$$\Phi(t) = \frac{1}{\omega}\begin{bmatrix} 4\sin\theta - 3\theta & -2(1-\cos\theta) \\ 2(1-\cos\theta) & \sin\theta \end{bmatrix} \tag{2}$$

where ω is the orbital frequency (radians per second) and θ the geocentric angle ωt. The coordinate frame used here is a Cartesian one, with the x axis horizontally forward along the orbit and the

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y axis vertically upward. It has been pointed out[2] that the motion described by Eq. (1), for any initial condition v_0, consists of an ellipse at the orbital frequency whose major axis is horizontal and twice the length of the minor axis, plus a horizontal drift at a rate of $-3\dot{x}_0$. The sense of the motion around the ellipse is in the forward direction on the lower side.

The in-plane dispersion of the fragments can be described by their 2×2 covariance matrix

$$P(t) = \overline{r(t)r^T(t)} \tag{3}$$

which, from Eq. (1), is given by

$$P(t) = \Phi(t)P_{v0}\Phi^T(t) \tag{4}$$

where P_{v0} is the covariance matrix of the initial breakup velocity vector. For the case of an isotropic breakup with velocity variance σ_v^2, we have $P_{v0} = \sigma_v^2 I$, so that

$$P(t) = \sigma_v^2 \Phi(t)\Phi^T(t) \tag{5}$$

The in-plane pinch points are the points where the determinant of P vanishes, which is equivalent to the vanishing of the determinant of Φ, given by

$$\begin{aligned}\omega^2 \det(\Phi) &= (4\sin\theta - 3\theta)\sin\theta + 4(1-\cos\theta)^2 \\ &= 8(1-\cos\theta) - 3\theta\sin\theta \end{aligned} \tag{6}$$

The behavior of $\det(\Phi)$ is shown in Fig. 1. It vanishes after each full orbit, as expected, but it also vanishes near the half-orbit points (except the first), i.e., at 1.407 orbits, 2.445, 3.461, etc. At these latter points, the fragment spread vanishes in a direction that is neither vertical nor horizontal. Thus, these pinch points cannot be detected, for example, by simply calculating the variances of the dispersion in the x and y directions (the diagonal elements of P).

At such a pinch point, the direction $\boldsymbol{e}$ in which the dispersion vanishes is such that $\boldsymbol{r}^T\boldsymbol{e} = 0$, or $\Phi^T\boldsymbol{e} = \boldsymbol{0}$. In other words, the vector $\boldsymbol{e}$ is a null vector of Φ^T, or an eigenvector of Φ^T corresponding to a zero eigenvalue. When Eq. (6) vanishes, it can be shown that $\Phi^T\boldsymbol{e} = \boldsymbol{0}$ if $\boldsymbol{e}$ is in the direction of

$$\boldsymbol{e} = [4 \quad 3\theta]^T \tag{7}$$

The in-plane dispersion, represented by the covariance matrix P, can be described in terms of a one-sigma dispersion ellipse (Fig. 2) whose semiaxes are the square roots of the eigenvalues of P. If P is written as

$$P = \begin{bmatrix} P_{xx} & P_{xy} \\ P_{xy} & P_{yy} \end{bmatrix} \tag{8}$$

then the two principal standard deviations are given by

$$\sigma_{\max}, \sigma_{\min} = \left(1/\sqrt{2}\right)\left[P_{xx} + P_{yy} \pm \sqrt{(P_{xx}-P_{yy})^2 + 4P_{xy}^2}\right]^{\frac{1}{2}} \tag{9}$$

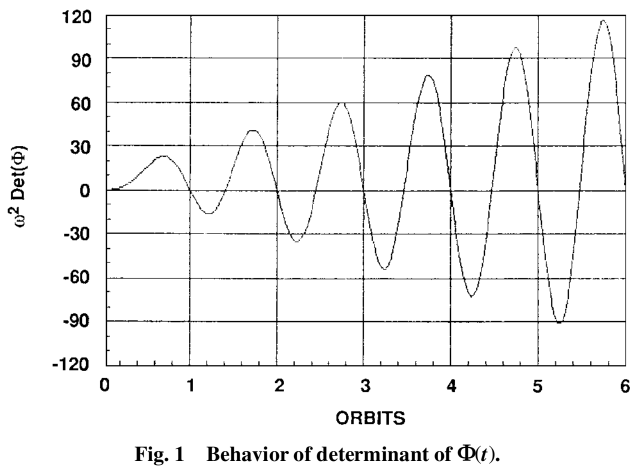

Fig. 1 Behavior of determinant of $\Phi(t)$.

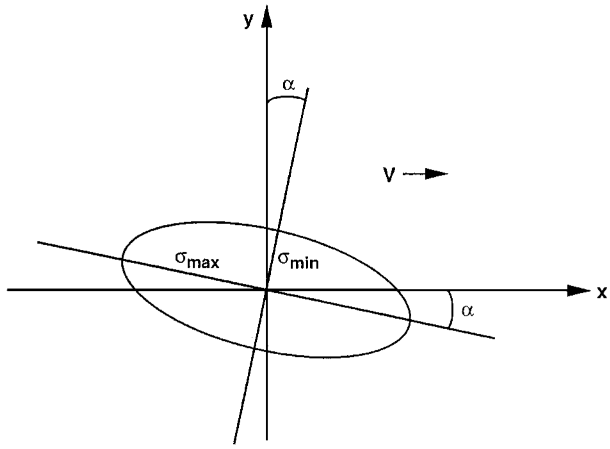

Fig. 2 Geometry of the dispersion ellipse.

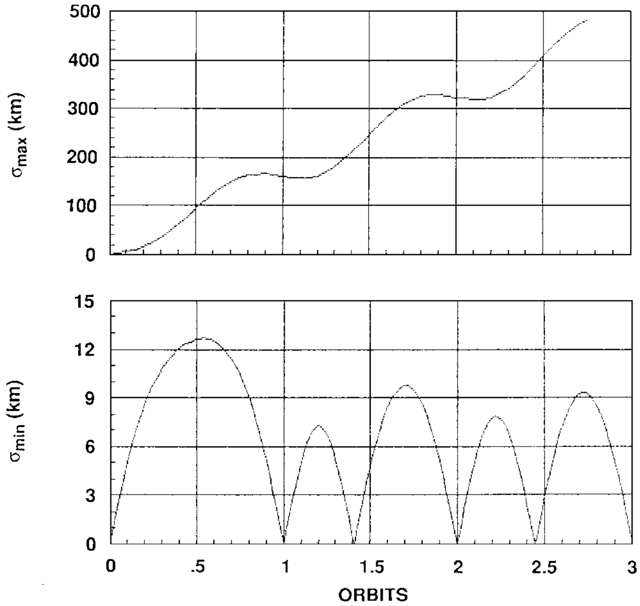

Fig. 3 Time histories of principal dispersions $\sigma_{\max}$ and $\sigma_{\min}$, for breakup velocity (standard deviation) σ_v = 10 m/s.

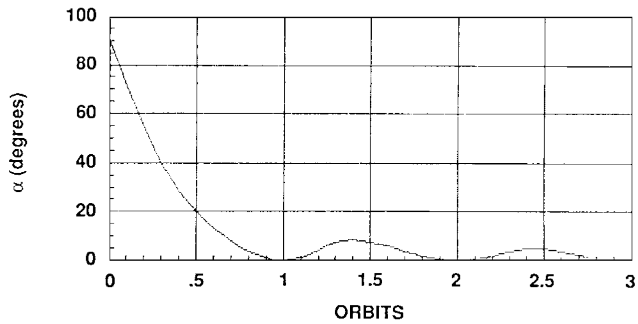

Fig. 4 Time history of dispersion-ellipse orientation angle.

The angle α, between the minor axis and the vertical direction, is given by

$$\alpha = \tfrac{1}{2}\text{ATAN2}(2P_{xy}, P_{yy} - P_{xx}) \tag{10}$$

where ATAN2 is a four-quadrant arctangent function as in Fortran. The behavior of these quantities is shown in Figs. 3 and 4, where we can observe the vanishing of $\sigma_{\min}$ at the supplementary pinch points and the corresponding nonzero values of α.

The α history of Fig. 4 agrees very closely with orientation data from actual observations of debris from the April 1975 breakup of the Cosmos 699 satellite.[5]

An interesting aspect of these supplementary pinch points is the way they are affected by small amounts of atmospheric drag on the fragments. It can be shown that a small constant drag force d on a fragment adds to Eq. (1) the term

$$\boldsymbol{r}_d(t) = \frac{d}{\omega^2}\begin{bmatrix} 3\theta^2/2 - 4(1-\cos\theta) \\ 2(\sin\theta - \theta) \end{bmatrix} \tag{11}$$

It is easy to show that at the supplementary pinch point, i.e., when Eq. (6) vanishes, $\boldsymbol{r}_d$ is orthogonal to the vector $\boldsymbol{e}$ of Eq. (7), so that these pinch points still remain, even though the full-orbit pinch points are dispersed in altitude by such drag forces. This is strictly true only if the drag force is effectively time invariant, which implies constant atmospheric density and rapidly tumbling fragments; however, it indicates a tendency that will still exist in more general cases, in that the average drag force will affect the orbits as described.

The existence of these supplementary pinch points does not seem to have been generally appreciated. Jehn[1] noticed a "near-singularity at 1.4 revolutions" in the volume of the debris cloud but erroneously attributed it to eccentricity effects, whereas Ashenberg[6] says that the zeros of the quantity in Eq. (6) "have no physical reason (these are a by-product of the linearization)." However, the identical phenomenon has been pointed out in studies of interplanetary guidance by Stern,[7] who refers to it as the "third singularity." In either context, a point on the orbit is singular with respect to an earlier reference point, if there exists a direction in which no position perturbation can be caused by any small velocity perturbation at the reference point. Stern shows that, for a general elliptical orbit, the condition for the third singularity is the vanishing of the quantity

$$X = [3(\Delta E/2) - e\sin(\Delta E/2)\cos E_{\text{av}}] \times [\cos(\Delta E/2) + e\cos E_{\text{av}}] - 4\sin(\Delta E/2) \tag{12}$$

where e is eccentricity, $\Delta E = E_2 - E_1$ is the difference in eccentric anomaly from the reference point to the point in question, and E_{av} is the average eccentric anomaly $(E_1 + E_2)/2$. For a circular orbit, it can be shown that the quantity in Eq. (6) can be written as

$$\omega^2 \det(\Phi) = -4\sin(\theta/2)\cdot X_0(\theta) \tag{13}$$

where $X_0(\theta)$ is the value of X with $e = 0$, and the difference in true anomaly θ is now identical to ΔE. Thus, the determinant vanishes (causing an in-plane pinch point) when $\theta = 2n\pi$ (n an integer, the full-orbit pinch point) and whenever $X_0(\theta)$ vanishes (the supplementary pinch point).

The supplementary pinch point is a planar one, in a plane normal to the nominal orbit plane, occurring just before a half-orbit point. It is followed almost immediately at the half-orbit point by another planar pinch point in the orbit plane. The combination, therefore, is very similar in character to a linear pinch point at midorbit (albeit oriented at some arbitrary angle). To the extent that drag forces are time invariant, they will allow this character to persist while changing the full-orbit pinch points from linear ones to planar ones; this is a complete reversal of the pinch-point characteristics as customarily envisioned. Whether these behaviors can actually be observed in practice is open to question because nonlinearities and other extraneous effects tend to cause deviations from our ideal linearized model after a few orbits, especially if breakup velocities are large.

Conclusions

We have demonstrated, using a linearized model of orbital perturbations, the existence of supplementary pinch points in the behavior of breakup fragments, analogous to the third singularity previously shown to exist in the context of interplanetary guidance. Small drag forces, to the extent that they are time invariant, will not cause these supplementary pinch points to vanish with time.

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